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NATURAL CONVECTION AROUND A SEMI-INFINITE VERTICAL PLATE: HIGHER-ORDER EFFECTS

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Abstract—On the basis of the first two terms in the boundary-layer expansion, it is shown by means of a global energy-rate balance that the leading-edge effect upon the total heat-transfer rate is given by $a_1 k\Delta T$ where a_1 has the value 0·625 at a Prandtl number of 0·72. As a result, the average Nusselt number for the semi-infinite plate is given by (for $\sigma = 0{\cdot}72$):

$$\bar{N} = 0{\cdot}476\, G^{1/4} + 0{\cdot}625 + O(G^{-1/12}) \qquad (1)$$

where the $O(G^{-1/12})$ term is indeterminate, arising from the appearance of an eigenfunction in the boundary-layer expansion.

Direct comparison of (1) with experimental data is precluded by the necessity of first determining finite-plate effects which, on the basis of recent analyses of the trailing-edge region for the forced-flow problem, are expected to contribute a term of $O(G^{1/16})$ to (1).

NOMENCLATURE

D, total drag on plate between leading edge and local x (per side and per unit width);
g, gravitational acceleration (along negative x-axis);
G, local Grashof number, $\equiv g\beta\Delta T x^3/\nu^2$;
L, length of plate (if finite);
N, local Nusselt number, $\equiv q_w x/(k\Delta T)$;
$\bar{N}$, average Nusselt number, $\equiv \dot{Q}/(k\Delta T)$;
q_w, local heat-transfer flux from plate, $\equiv -k(\partial T/\partial y)_{y=0}$;
$\dot{Q}$, total heat-transfer rate from plate between leading edge and local x (per side and per unit width);
r, polar radial coordinate;
T, temperature;
u, x-component of velocity;
U, $\equiv (\nu/x)G^{1/2}$;
U_∞, magnitude of uniform stream in forced-flow case;
v, y-component of velocity;
x, vertical coordinate;
y, horizontal coordinate.

Greek symbols

β, coefficient of thermal expansion;
δ, characteristic thickness of boundary layer, $\equiv x/G^{1/4}$;
ΔT, $\equiv T_w - T_\infty$;
ε, $\equiv \delta/x \equiv G^{-1/4} \equiv (x/\lambda)^{-3/4}$;
η, $\equiv y/\delta$;
θ, angular coordinate measured from plate;
λ, $\equiv (\nu^2/(g\beta\Delta T))^{1/3}$;
μ, dynamic viscosity;
ν, kinematic viscosity;
ρ, density of fluid;
σ, Prandtl number.

Subscripts

L, value at $x = L$;
W, value at wall;
∞, value of undisturbed state.

1. INTRODUCTION

HIGHER-ORDER boundary-layer effects for natural convection around a semi-infinite vertical plate have been considered by Yang and Jerger [1] and Kadambi [2]. In both cases, the boundary-layer expansion was in powers of $G^{-1/4}$. The first-order correction to the classical solution was obtained in [1] and this was extended to second-order in [2]. Although the first-order correction to the temperature field in the boundary layer was found to be identically zero, neither of the above investigations noted that the first-order correction to the global heat transfer is *not* zero. More specifically, by calculating the total thermal convection at any value of x in the boundary-layer region, it is shown in the present paper that the first-order correction to the global heat transfer (per side and unit width of plate) is given by $a_1 k\Delta T$ where a_1 is a Prandtl-number-dependent numerical factor having the value 0·6248 at $\sigma = 0{\cdot}72$. It is noted that this contribution is not x-dependent and therefore does not arise from a local conductive heat transfer from the plate in the boundary-layer region (hence, it went unnoticed in the previous investigations) but, rather, is to be interpreted as the "leading-edge" effect. (The "leading-edge region" corresponding to $r = O(\lambda)$ and representing the region in which the boundary-layer approximation breaks down.) It is remarkable that the boundary-layer expansion contains an explicit evaluation of the total heat-transfer contribution by a region in which the boundary-layer approximation is itself non-applicable. This result is completely analogous to that of Imai [3] who, by means of a global force balance, determined that the leading-edge drag is $1{\cdot}163\mu U_\infty$ for forced flow over an aligned semi-infinite plate.

Although not considered in the previous investigations, the present paper also includes a determination of the first three eigenvalues and eigenfunctions appearing in the boundary-layer expansion. The most important of these corresponds to a correction of order $(G^{-1/4})^{4/3}$ and therefore appears before the second-order term obtained in [2]. In the manner of Stewartson [4], this leading indeterminacy can be interpreted as an apparent shift in the location of the leading edge of plate as seen from the boundary-layer region.

It is also shown in the present paper that the second-order correction obtained in [2] is incorrect. In particular, the error is due to improper matching considerations.

2. BOUNDARY-LAYER EXPANSION

Employing the Boussinesq approximation and neglecting viscous dissipation, the governing equations are:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2.1}$$

$$\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)u = \nu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u + g\beta(T-T_\infty)-\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{2.2}$$

$$\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)v = \nu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)v-\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{2.3}$$

$$\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)T = \frac{\nu}{\sigma}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)T \tag{2.4}$$

subject to the boundary conditions:

$$u = 0 = v,\quad T = T_w \quad \text{at} \quad y = 0,\quad x > 0 \tag{2.5}$$

$$\frac{\partial u}{\partial y} = 0 = v,\quad \frac{\partial T}{\partial y} = 0 \quad \text{at} \quad y = 0,\quad x < 0 \tag{2.6}$$

$$u \sim 0 \sim v,\quad T \sim T_\infty \quad \text{as} \quad r \to \infty,\quad \theta \neq 0. \tag{2.7}$$

If possible indeterminacies (eigenfunctions) are temporarily omitted (to be considered in Section 4) then appropriate expansions for the streamfunction, temperature and pressure in the boundary-layer region,

$$y = O(\delta),\quad x > O(\lambda),$$

are given by:

$$\psi = U\delta\{F_0(\eta)+\varepsilon F_1(\eta)+\varepsilon^2 F_2(\eta)+\ldots\} \tag{2.8}$$

$$T = T_\infty + \Delta T\{H_0(\eta)+\varepsilon H_1(\eta)+\varepsilon^2 H_2(\eta)+\ldots\} \tag{2.9}$$

$$p = \rho U^2\{\varepsilon^2 P_2(\eta)+\ldots\} \tag{2.10}$$

where F_0 and H_0 correspond to the classical problem:

$$\left.\begin{aligned} F_0'''+\tfrac{3}{4}F_0F_0''-\tfrac{1}{2}F_0'F_0'+H_0 &= 0\\ H_0''+\tfrac{3}{4}\sigma F_0 H_0' &= 0\\ F_0(0) = 0 = F_0'(0) = F_0'(\infty) &= H_0(\infty),\\ H_0(0) &= 1. \end{aligned}\right\} \tag{2.11}$$

The boundary-layer region is complemented by an irrotational, isothermal, external region:

$$\theta \neq 0,\quad r > O(\lambda)$$

where appropriate expansions are:

$$\psi = \tilde\psi_1+\tilde\psi_2+\ldots \tag{2.12}$$

$$T = T_\infty \tag{2.13}$$

$$p = \tilde p_1+\tilde p_2+\ldots. \tag{2.14}$$

In particular, numerical integration of (2.11) for $\sigma = 0{\cdot}72$ results in:

$$F_0''(0) = 0{\cdot}95604,\quad H_0'(0) = -0{\cdot}35683,\quad F_0(\infty) = 1{\cdot}69389. \tag{2.15}$$

If $F_0(\infty)$ is denoted by "A_0" then, as shown by Yang and Jerger [1], $\tilde\psi_1$ is governed by

$$\nabla^2\tilde\psi_1 = 0;\quad \tilde\psi_1)_{\theta=0} = A_0\nu\left(\frac{r}{\lambda}\right)^{3/4},\quad \tilde\psi_1)_{\theta=\pi} = 0 \tag{2.16}$$

where the governing equation indicates that the vorticity is zero, and the inhomogeneous boundary condition represents a matching of v with the boundary-layer solution. The solution of (2.16) can be written as

$$\tilde\psi_1 = -\sqrt{2}A_0\nu\left(\frac{r}{\lambda}\right)^{3/4}\sin\tfrac{3}{4}(\theta-\pi) \tag{2.17}$$

which, when combined with Bernoulli's equation, results in

$$\tilde{p}_1 = -\left(\frac{3}{4}A_0\right)^2 \rho\left(\frac{\nu}{\lambda}\right)^2\left(\frac{\lambda}{r}\right)^{1/2}. \quad (2.18)$$

By expanding (2.17) and (2.18) around $\theta = 0$ and noting that $\theta = \varepsilon\eta + O(\varepsilon^3)$, $r = x[1+O(\varepsilon^2)]$, it follows that the behavior of $\tilde{\psi}_1$ and $\tilde{p}_1$ in the matching region is given by

$$\tilde{\psi}_1 = A_0 U\delta[1+\tfrac{3}{4}\varepsilon\eta + O(\varepsilon^2)] \quad (2.19)$$

$$\tilde{p}_1 = -(\tfrac{3}{4}A_0)^2\rho\varepsilon^2 U^2[1+O(\varepsilon^2)]. \quad (2.20)$$

Hence, as shown by Yang and Jerger [1], F_1 and H_1 are governed by

$$\left.\begin{aligned} F_1''' + \tfrac{3}{4}F_0F_1'' - \tfrac{1}{4}F_0'F_1' + H_1 &= 0 \\ H_1'' + \tfrac{3}{4}\sigma(F_0H_1)' &= 0 \\ F_1(0) = 0 = F_1'(0) = H_1(0) &= H_1(\infty), \\ F_1'(\infty) &= \tfrac{3}{4}A_0 \end{aligned}\right\} \quad (2.21)$$

where the inhomogeneous boundary condition represents a matching with the $O(\varepsilon)$ term in (2.19). Numerical integration of (2.21) at $\sigma = 0{\cdot}72$ results in

$$F_1''(0) = 0{\cdot}39636, \quad H_1 \equiv 0, \quad A_1 = -2{\cdot}53796 \quad (2.22)$$

where

$$F_1(\eta) \sim \tfrac{3}{4}A_0\eta + A_1 \quad \text{as} \quad \eta \to \infty. \quad (2.23)$$

It follows that $\tilde{\psi}_2$ is governed by

$$\nabla^2\tilde{\psi}_2 = 0; \quad \tilde{\psi}_2)_{\theta=0} = A_1\nu, \quad \tilde{\psi}_2)_{\theta=\pi} = 0 \quad (2.24)$$

which, as shown by Kadambi [2], results in

$$\tilde{\psi}_2 = A_1\nu\left(1 - \frac{\theta}{\pi}\right). \quad (2.25)$$

Combined with $\tilde{\psi}_1$ and Bernoulli's equation, it follows that

$$\tilde{p}_2 = \frac{3\sqrt{2}}{4\pi}A_0A_1\rho\left(\frac{\nu}{\lambda}\right)^2\left(\frac{\lambda}{r}\right)^{5/4}\cos\tfrac{3}{4}(\theta - \pi). \quad (2.26)$$

Finally, the governing equations for P_2, F_2 and H_2 are given by

$$P_2' = \tfrac{1}{16}\eta F_0'F_0' + \tfrac{3}{16}\eta F_0F_0'' - \tfrac{9}{16}F_0F_0' + \tfrac{1}{4}\eta F_0''' - \tfrac{1}{4}F_0'' \quad (2.27)$$

$$\begin{aligned} F_2''' + \tfrac{3}{4}F_0F_2'' + \tfrac{1}{2}F_0'F_2' - \tfrac{3}{4}F_0''F_2 + H_2 &= -\tfrac{1}{2}P_2 \\ -\tfrac{1}{4}\eta P_2' - \tfrac{1}{4}F_1'F_1' + \tfrac{1}{4}F_0' - \tfrac{1}{16}\eta F_0'' &- \tfrac{1}{16}\eta^2F_0''' \end{aligned} \quad (2.28)$$

$$\begin{aligned} H_2'' + \sigma(\tfrac{3}{4}F_0H_2' + \tfrac{1}{2}F_0'H_2 - \tfrac{3}{4}H_0'F_2) \\ = -\tfrac{5}{16}\eta H_0' - \tfrac{1}{16}\eta^2H_0'' \end{aligned} \quad (2.29)$$

subject to the conditions:

$$\left.\begin{aligned} F_2(0) = 0 = F_2'(0) = H_2(0) &= H_2(\infty), \\ P_2(\infty) &= -(\tfrac{3}{4}A_0)^2, \\ F_2(\eta) \sim \tfrac{3}{32}A_0\eta^2 - \frac{A_1}{\pi}\eta + A_2 \quad &\text{as} \quad \eta \to \infty. \end{aligned}\right\} \quad (2.30)$$

It is noted that (2.27)–(2.29) are in agreement with Kadambi [2] but that (2.30) is not. In particular, the values of $P_2(\infty)$ and $F_2''(\infty)$ are specified incorrectly in [2]; specifically, the non-zero values of $P_2(\infty)$ and $F_2''(\infty)$ in (2.30) follow from the behavior of $\tilde{p}_1$ in (2.20) and the $O(\varepsilon^2)$ term in (2.19), respectively. Numerical integration at $\sigma = 0{\cdot}72$ results in

$$F_2''(0) = -0{\cdot}99487, \quad H_2'(0) = -0{\cdot}89145, \quad A_2 = -1{\cdot}50005. \quad (2.31)$$

3. GLOBAL CONSIDERATIONS

Based upon the results in Section 2, it follows that the local wall heat flux in the boundary-layer region is given by

$$q_w = -k\frac{\Delta T}{\delta}\{H_0'(0) + \varepsilon^2H_2'(0) + O(\varepsilon^3)\}. \quad (3.1)$$

Although the total heat transfer could be obtained by evaluating the quantity

$$\int_0^x q_w \, dx,$$

(3.1) is only applicable where $\varepsilon < O(1)$, i.e. $x > O(\lambda)$, as is suggested by the fact that the second term in (3.1) is not integrable at $x = 0$. Alternatively, the global heat transfer can be obtained by calculating the total thermal convection at any cross-section of the boundary layer. That is,

$$\begin{aligned} \dot{Q} = \rho C_p \int_{\text{B.L.}} u(T - T_\infty)\,dy &= \rho C_p U\Delta T\delta\int_0^\infty F'H\,d\eta \\ &= \frac{k\Delta T}{\varepsilon}\{a_0 + a_1\varepsilon + a_2\varepsilon^2 + O(\varepsilon^3)\} \end{aligned} \quad (3.2)$$

where

$$a_0 = \sigma\int_0^\infty F_0'H_0\,d\eta, \quad a_1 = \sigma\int_0^\infty F_1'H_0\,d\eta,$$
$$a_2 = \sigma\int_0^\infty (F_0'H_2 + F_2'H_0)\,d\eta. \quad (3.3)$$

In particular, for $\sigma = 0{\cdot}72$,

$$a_0 = 0{\cdot}47577, \quad a_1 = 0{\cdot}62480, \quad a_2 = -1{\cdot}18859. \quad (3.4)$$

Hence, it is seen from (3.1) and (3.2) that although the $O(\varepsilon)$ correction to q_w vanishes in the boundary-layer region, the $O(\varepsilon)$ correction for $\dot{Q}$ is non-zero and, from (3.3), is reflected in the boundary layer as a convection of the primary temperature by the first-order velocity. This energy must arise from the heat transfer in the leading-edge region and indicates that the local behavior of q_w in the latter region must exceed that given by the leading term in (3.1). That is, letting

$$q_{w_0} \equiv -k\frac{\Delta T}{\delta}H_0'(0) \quad (3.5)$$

and noting that

$$a_0 \frac{k\Delta T}{\varepsilon} = \int_0^x q_{w_0}\,dx \qquad (3.6)$$

it follows from (3.2) that

$$a_1 k\Delta T = \lim_{\left(\frac{x}{\lambda}\right)\to\infty} \int_0^x (q_w - q_{w_0})\,dx. \qquad (3.7)$$

That is, the l.h.s. of (3.7) is equal to the integrated value of the difference between the actual wall heat flux and that corresponding to the classical boundary-layer solution. In other words, $a_1 k\Delta T$ represents the leading-edge effect upon the global heat-transfer rate.

In a similar manner, a global force balance in the vertical direction gives the following result for the total drag on plate (per side and unit width) between the leading edge and local value of x:

$$D = \rho\frac{\nu^2}{\lambda}\{\varepsilon^{-5/3}[\tfrac{4}{5}F_0''(0)] + \varepsilon^{-2/3}[2F_1''(0)] + O(1) + \varepsilon^{1/3}[-4F_2''(0)] + \ldots\} \qquad (3.8)$$

where the $O(1)$ contribution is indeterminate and is due to the global buoyant force acting throughout the leading-edge region or, alternatively, can be interpreted as the leading-edge effect upon the total drag. That is, unlike the forced-flow problem considered by Imai [3], the boundary-layer expansion in the present problem does *not* contain an explicit evaluation of the leading-edge drag.

4. EIGENVALUES-EIGENFUNCTIONS

If the eigenvalues associated with the boundary-layer expansion are denoted by "α_n" and the associated normalized eigenfunction by $\phi_n(\eta)$, $\theta_n(\eta)$, then additional terms of the form $U\delta\varepsilon^{\gamma_n}C_n\phi_n$ and $\Delta T\varepsilon^{\gamma_n}C_n\theta_n$ must be added to the expansions in (2.8) and (2.9), respectively, with the multiplicative constant C_n being indeterminate, in general.

Inclusion of these terms in the boundary-layer expansion and substitution into (2.2) and (2.4) results in the following linear homogeneous equations for ϕ_n and θ_n:

$$\phi_n''' + \tfrac{3}{4}F_0\phi_n'' + (\tfrac{3}{4}\alpha_n - 1)F_0'\phi_n' + \tfrac{3}{4}(1-\alpha_n)F_0''\phi_n + \theta_n = 0 \qquad (4.1)$$

$$\frac{1}{\sigma}\theta_n'' + \tfrac{3}{4}F_0\theta_n' + \tfrac{3}{4}\alpha_n F_0'\theta_n + \tfrac{3}{4}(1-\alpha_n)H_0'\phi_n = 0 \qquad (4.2)$$

subject to the homogeneous boundary conditions:

$$\phi_n(0) = 0 = \phi_n'(0) = \phi_n'(\infty) = \theta_n(0) = \theta_n(\infty). \qquad (4.3)$$

There exist non-trivial solutions of (4.1)–(4.3) only for particular values of α_n. In particular, the smallest such value is $\alpha_1 = 4/3$ with corresponding eigenfunction given by

$$\phi_1 = \frac{1}{A_0}(F_0 - \tfrac{1}{3}\eta F_0'), \quad \theta_1 = -\frac{1}{3A_0}\eta H_0' \qquad (4.4)$$

where the normalization has been taken to be $\phi_n(\infty) = 1$. Hence, for $\sigma = 0{\cdot}72$,

$$\phi_1''(0) = 0{\cdot}18818, \quad \theta_1'(0) = 0{\cdot}07022. \qquad (4.5)$$

By means of a straightforward numerical scheme, it is found that, for $\sigma = 0{\cdot}72$:

$$\left.\begin{aligned} \alpha_2 &= 3{\cdot}189, \quad \phi_2''(0) = -1{\cdot}082, \quad \theta_2'(0) = -0{\cdot}2245 \\ \alpha_3 &= 8{\cdot}311, \quad \phi_3''(0) = 0{\cdot}2160, \quad \theta_3'(0) = 0{\cdot}4602. \end{aligned}\right\} \qquad (4.6)$$

It is noted that $U\delta\varepsilon^{4/3}\phi_1$ and $\Delta T\varepsilon^{4/3}\theta_1$ are proportional to the x-derivative of the zeroth-order streamfunction and temperature in (2.8) and (2.9). Hence, following Stewartson [4], C_1 is related to an apparent shift in the location of the leading-edge as seen by the boundary-layer region.

It is also noted that the contribution to $\dot{Q}$ by the leading eigenfunction is

$$C_1 a_{4/3} k\Delta T\varepsilon^{1/3} \qquad (4.7)$$

where

$$a_{4/3} = \sigma\int_0^\infty (F_0'\theta_1 + \phi_1' H_0)\,d\eta \qquad (4.8)$$

which takes on the value 0·28087 at $\sigma = 0{\cdot}72$.

On the other hand, the contribution of the leading eigenfunction to D is given by

$$4C_1\rho\frac{\nu^2}{\lambda}\varepsilon^{-1/3}\phi_1''(0) \qquad (4.9)$$

which is seen to be unbounded as $x/\lambda \to \infty$. That is, although the leading eigenfunction results in a negligible contribution to the global heat transfer in the boundary-layer region, the interaction of $C_1\Delta T\varepsilon^{4/3}\theta_1$ with the gravitational field results in a buoyant force which, when integrated over the boundary layer, is proportional to $(x/\lambda)^{1/4}$. Such an unbounded contribution to D by an eigenfunction is apparently peculiar to natural-convection boundary layers, i.e. to flows in which a body force acts throughout the boundary-layer region.

5. DISCUSSION

Based upon the preceding sections, it follows that the local wall heat flux in the boundary-layer region is given by (for $\sigma = 0{\cdot}72$):

$$q_w = \frac{k\Delta T}{\delta}\{0{\cdot}3568 - 0{\cdot}0702\,C_1G^{-1/3} + 0{\cdot}8915\,G^{-1/2} + O(G^{-2/3})\} \qquad (5.1)$$

where the bracketed quantity also equals $N/G^{1/4}$ and the $O(G^{-2/3})$ term is due to the non-linear interaction of the leading eigenfunction with itself. Similarly, the total heat transfer is given by:

$$\dot{Q} = k\Delta T G^{1/4}\{0{\cdot}4758+0{\cdot}6248\,G^{-1/4} +0{\cdot}2809\,C_1 G^{-1/3}-1{\cdot}1886\,G^{-1/2}+O(G^{-2/3})\} \quad (5.2)$$

where the bracketed quantity also equals $\bar{N}/G^{1/4}$. Hence, although the leading correction to the local Nusselt number is proportional to C_1 and is therefore indeterminate, the leading correction to the average Nusselt number is known explicitly. That is,

$$\bar{N} = 0{\cdot}476\,G^{1/4}+0{\cdot}625+O(G^{-1/12}). \quad (5.3)$$

Clearly, it would be desirable to compare (5.3) with available experimental results. However, all such data for $\bar{N}$ are averaged over the entire length of a finite plate. Hence, in addition to basing $\bar{N}$ and G in (5.3) upon $x = L$, it would also be necessary to include the effects of the trailing edge and wake region upon $\bar{N}$.

Yang and Jerger [1] have indeed estimated the effect of the wake upon the boundary-layer and their result would require subtracting $\approx 0{\cdot}325$ from the 0·625 in (5.3). However, a more accurate estimate in the appendix indicates that 0·086, rather than 0·325, should be subtracted.

A second effect arising from the finite plate length is the breakdown of the boundary-layer approximation in the vicinity of the trailing edge. In a manner completely analogous to that of Imai [5] and Stewartson [6] for the forced-flow problem, it follows that the trailing-edge region is of order $L/G_L^{3/8}$ in extent and that it contributes a term of $O(G_L^{-1/8})$ in (5.3). However, in a revised analysis by Stewartson [7] and an independent investigation by Messiter [8], it has been shown on an order-of-magnitude basis that the effect of the trailing edge is more complicated and extensive than above. According to this more recent "triple-deck" analysis, the influence of the trailing edge in the present problem extends over a region of order $L/G_L^{3/16}$ and contributes a term of $O(G_L^{1/16})$ to (5.3). That is, the triple-deck theory indicates that the trailing-edge effect upon $\bar{N}$ is larger than that of the leading edge.

If this latter structure is correct, then, pending its detailed numerical solution, the leading correction to $\bar{N}$ for the finite plate remains unknown and comparison with experiment is unwarranted.

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APPENDIX

Interaction of Wake and Boundary Layer (Finite Plate)

This effect arises from the influence of the wake upon the irrotational-flow region, particularly upon the velocity at the outer edge of the boundary layer. In estimating this effect, Yang and Jerger [1] followed a procedure employed by Kuo [9] for the forced-flow problem and obtained the first-order irrotational flow by integrating a variable sink distribution along $y = 0$ based upon $v = 0$ for $x < 0$, $v = \mp\frac{3}{4}A_0\varepsilon U$ for $0 \leqslant x \leqslant L(y = 0^{\pm})$ and $v = 0$ for $x > L$.

It is noted, however, that whereas the asymptotic wake in the forced-flow problem corresponds to the flow downstream of a concentrated drag force, the solution of which has zero influx at its outer edge, the asymptotic wake in the present problem corresponds to the natural-convection plume above a concentrated heat source, the solution of which has a non-zero influx. In particular, if the location of the heat source is taken to be the trailing edge of the plate then (see, e.g. Fujii [10]), at the outer edge of the plume (for $\sigma = 0{\cdot}72$):

$$v = \mp 1{\cdot}310\frac{\nu}{\tilde{\lambda}}\left(\frac{\tilde{x}}{\tilde{\lambda}}\right)^{-2/5} \quad (A.1)$$

where $\tilde{\lambda} \equiv \nu^2/(g\beta\tilde{\Delta}T)^{1/3}$, $\tilde{\Delta}T \equiv \dot{Q}/k$, $\tilde{x} \equiv x-L$ and $\dot{Q} \approx 2\,(0{\cdot}476)k\Delta T G_L^{1/4}$ where the factor "2" is due to the two sides of the plate. Hence, (A.1) can be rewritten as:

$$v = \mp 1{\cdot}297\frac{\nu}{L}G_L^{1/4}\left(\frac{x-L}{L}\right)^{-2/5} \quad (A.2)$$

which is to be compared with v at the outer edge of boundary layer:

$$v = \mp 1{\cdot}270\frac{\nu}{L}G_L^{1/4}\left(\frac{x}{L}\right)^{-1/4} \quad (A.3)$$

where the numerical factor in (A.3) is the value of $\frac{3}{4}A_0$ at $\sigma = 0{\cdot}72$.

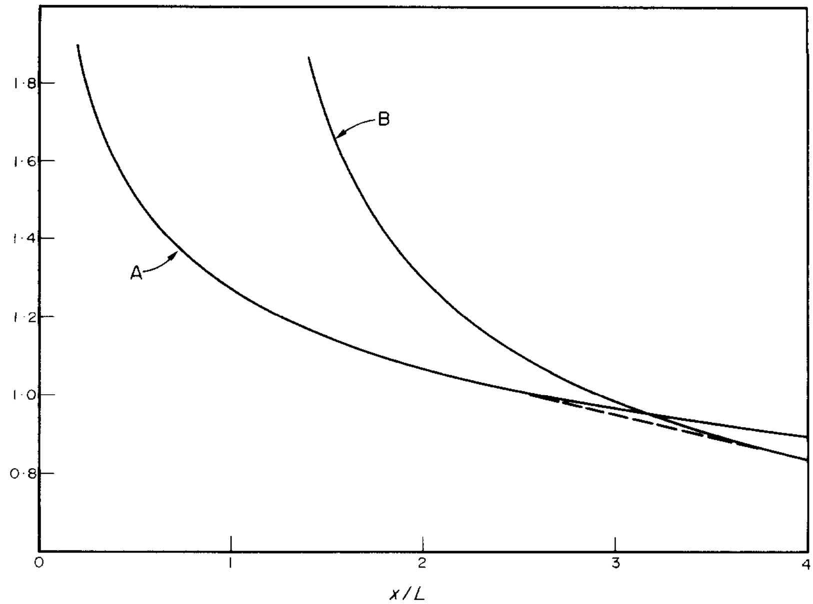

FIG. 1. Curve A: $1{\cdot}270(x/L)^{-1/4}$; curve B: $1{\cdot}297((x-L)/L)^{-2/5}$; dashed curve by graphical interpolation.

A plot of these two results, together with a curve fit connecting the two between their respective regions of applicability, is shown in Fig. 1. The difference between the $(x/L)^{-1/4}$ curve and the lower curve represents the effect of the wake. If this difference is denoted as $q(\xi)$, where $\xi \equiv x/L$, and if $u_c(\xi)$ denotes the wake-induced correction to u at the outer edge of boundary layer, then, by superposition of a variable sink distribution,

$$u_c(\xi) = -\frac{1}{\pi}\frac{\nu}{L}G_L^{1/4}\int_{2\cdot 4}^{\infty}\frac{q(t)\,\mathrm{d}t}{t-\xi}. \qquad \text{(A.4)}$$

The resulting values of u_c for $0 \leqslant \xi \leqslant 1$ can be fairly well approximated by

$$u_c(\xi) = -\frac{\nu}{L}G_L^{1/4}[0{\cdot}4580+0{\cdot}0189\xi+0{\cdot}0022\xi^2]. \qquad \text{(A.5)}$$

Hence, following Yang and Jerger [1] but with use of this more appropriate u_c, the resulting contribution to $\dot{Q}$ by the wake is $\approx -0{\cdot}086\,k\Delta T$ at $\sigma = 0{\cdot}72$. (It is noted that the same value was obtained when the heat source was taken to be at $\xi = 0{\cdot}6$ rather than $\xi = 1$.)

CONVECTION NATURELLE SUR UNE PLAQUE VERTICALE SEMI-INFINIE: EFFETS D'ORDRE ELEVE

Résumé—En considérant les deux premiers termes du développement de la couche limite, on montre à l'aide du bilan global d'énergie, que l'effet du bord d'attaque sur le taux de transfert thermique est donné par $a_1 k\Delta T$, où a_1 prend la valeur 0,625 pour un nombre de Prandtl égal à 0,72. Le nombre de Nusselt moyen, pour la plaque semi-infinie, est donné par ($\sigma = 0{,}72$):

$$\bar{N} = 0{,}476G^{1/4}+0{,}625+O(G^{-1/12}) \qquad (1)$$

où le terme indéterminé $O(G^{-1/12})$ s'introduit sous l'aspect d'une fonction propre dans le développement de couche limite.

Une comparaison directe de (1) avec les résultats expérimentaux est difficile à cause de la nécessité de définir en premier les effets d'une longueur finie qui, sur la base d'analyses récentes de la région du bord de fuite pour la convection forcée, sont supposés faire intervenir dans (1) un terme $O(G^{1/16})$.

FREIE KONVEKTION UM EINE HALBUNENDLICHE, VERTIKALE PLATTE: EINFLÜSSE HÖHERER ORDNUNG

Zusammenfassung—Ausgehend von den ersten beiden Gliedern in der Gleichung für die Grenzschichtdicke wird mit Hilfe einer Gesamtenergiebilanz gezeigt, daß der Einfluß der Vorderkante auf den gesamten Wärmeübergang durch $a_1\,k\Delta T$ gegeben ist, wobei a_1 bei einer Prandtl-Zahl von 0,72 den Wert 0,625 hat.

Als Ergebnis wird eine mittlere Nusselt-Zahl für die halbunendliche Platte angegeben mit

$$\bar{N}u = 0{,}476\,G^{1/4} + 0{,}625 + O(G^{-1/12}) \qquad (1),$$

wobei $O(G^{-1/12})$, vom Auftreten einer Eigenfunktion in der Grenzschichtdicke an, unbestimmt ist.

Der direkte Vergleich von (1) mit experimentellen Ergebnissen kann erst nach Abschluß von Untersuchungen an endlichen Platten durchgeführt werden, doch ist aufgrund früherer Untersuchungen des Bereichs der Hinterkante bei erzwungener Strömung zu erwarten, daß bei diesen der Term $O(G^{1/16})$ zu (1) dazukommt.

ЕСТЕСТВЕННАЯ КОНВЕКЦИЯ ВБЛИЗИ ПОЛУБЕСКОНЕЧНОЙ ВЕРТИКАЛЬНОЙ ПЛАСТИНЫ. ЭФФЕКТЫ ВЫСШЕГО ПОРЯДКА

Аннотация — На основе первых двух членов в разложении пограничного слоя посредством вычисления общего энергетического балланса показано, что влияние передней кромки на суммарную интенсивность теплообмена определяется величиной $a_1 k \Delta T$, где a_1 равно 0,625 при $Pr = 0{,}72$. В результате среднее число Нуссельта для полубесконечной пластины при $\sigma = 0{,}72$ определяется следующим образом:

$$\bar{N} = 0{,}476G^{1/4} + 0{,}625 + O(G^{-1/12}) \qquad (1)$$

где $O(G^{-1/12})$ — неопределенный член, вытекающий вследствие появления собственной функции в разложении пограничного слоя.

Непосредственное сравнение уравнения (1) с экспериментальными данными затруднено прежде всего из-за необходимости определения эффектов конечной пластины, которые, как предполагается, на основе последних исследований области задней кромки в задаче о вынужденном течении приводят к появлению в уравнении (1) члена порядка $O(G^{1/16})$.